GCE: Analysis, measure theory, Lebesgue integration

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Exercise 1:

a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is bounded. Give an example, with proof, of such a function f whose improper Riemann integral on $(-\infty, \infty)$ exists and is finite, but which is not in $L^1(\mathbb{R})$.

b) Suppose $-\infty < a < b < \infty$. Prove that if the *proper* Riemann integral of a function g on $[a, b]$ exists, then the Lebesgue integral of g on $[a, b]$ exists and equals the value of the proper Riemann integral.

Exercise 2:

Let f_n be a sequence of measurable functions from $[0, 1]$ to $\mathbb{R}$. Assume that each function f_n is finite almost everywhere. Show that f_n converges in measure to zero if and only if

$$\lim_{n\to\infty} \int_0^1 \frac{|f_n|}{1 + |f_n|} = 0$$

Hint: Recall that by definition f_n *converges in measure* to f if and only if, given any $\epsilon > 0$,

$$\lim_{n\to\infty} |\{|f_n - f| > \epsilon\}| = 0\,.$$

Exercise 3:

a) Let $(X, \mathcal{A}, \mu)$ be a measure space, and f_n a converging sequence in $L^1(X)$. Show that f_n has a subsequence which is convergent almost everywhere.

b) Find a sequence g_n in $L^1([0, 1])$ such that: g_n converges in $L^1([0, 1])$ and for all x in [0,1] the sequence $g_n(x)$ diverges.

c) In the measure space $(X, \mathcal{A}, \mu)$, let A_n be a sequence of elements of $\mathcal{A}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$ and let f be in $L^1(X)$. Show that $\lim_{n\to\infty} \int_{A_n} f = 0$

Exercise 4:

Suppose $f \in L^1(\mathbb{R})$ is such that $f > 0$, almost everywhere. Show that $\int f > 0$.